

Observables as functions: Antonymous functions

Andreas Döring

IAMPh, Fachbereich Mathematik,
J. W. Goethe-Universität Frankfurt, Germany

adoering@math.uni-frankfurt.de

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Antonymous functions are real-valued functions on the Stone spectrum of a von Neumann algebra \mathcal{R} . They correspond to the self-adjoint operators in \mathcal{R} , which are interpreted as observables in quantum physics. Antonymous functions turn out to be generalized Gelfand transforms, related to de Groote's observable functions.

1 Introduction

The **Stone spectrum** $\mathcal{Q}(\mathcal{R})$ of a unital von Neumann algebra $\mathcal{R} \subseteq \mathcal{L}(\mathcal{H})$ is defined as the space of maximal filter bases¹ (or, equivalently, maximal dual ideals² [Bir73]) in the projection lattice $\mathcal{P}(\mathcal{R})$ of \mathcal{R} [deG05]. The sets

$$\mathcal{Q}_P(\mathcal{R}) := \{\mathfrak{B} \in \mathcal{Q}(\mathcal{R}) \mid P \in \mathfrak{B}\}, \quad P \in \mathcal{P}(\mathcal{R})$$

form the base of a topology on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ such that $\mathcal{Q}(\mathcal{R})$ becomes a zero-dimensional, completely regular Hausdorff space. The sets $\mathcal{Q}_P(\mathcal{R})$ are closed-open. If the von Neumann algebra \mathcal{R} is abelian, then the Stone spectrum $\mathcal{Q}(\mathcal{R})$ is homeomorphic to the Gelfand spectrum $\Omega(\mathcal{R})$ of \mathcal{R} . For an arbitrary non-abelian unital von Neumann algebra \mathcal{R} , the Stone spectrum can hence be regarded as a *non-commutative generalization of the Gelfand spectrum*. The elements \mathfrak{B} of the Stone spectrum $\mathcal{Q}(\mathcal{R})$ are called **quasipoints**.

¹A subset \mathcal{F} of elements of a lattice \mathbb{L} with zero element 0 is a **filter base** if (i) $0 \notin \mathcal{F}$ and (ii) for all $a, b \in \mathcal{F}$, there is a $c \in \mathcal{F}$ such that $c \leq a \wedge b$.

²A subset \mathcal{D} of elements of a lattice \mathbb{L} with zero element 0 is a **dual ideal** if (i) $0 \in \mathcal{D}$, (ii) $a, b \in \mathcal{D}$ implies $a \wedge b \in \mathcal{D}$ and (iii) $a \in \mathcal{D}$ and $b \geq a$ imply $b \in \mathcal{D}$.

In this article, we show that for each self-adjoint operator $A \in \mathcal{R}_{sa}$, there is a real-valued function $g_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ with the following properties: (i) the image of g_A is the spectrum of A , (ii) g_A is continuous and (iii) if \mathcal{R} is abelian, then g_A coincides with the Gelfand transform \widehat{A} of A . The function g_A , called the **function antonymous with A** (or the antonymous function of A), can hence be regarded as a *generalized Gelfand transform* of the self-adjoint operator $A \in \mathcal{R}_{sa}$.

Antonymous functions are related to *observable functions* introduced by de Groote and have analogous properties, compare [deG05b]. Together, these functions give a novel view on quantum observables. Each physical observable, represented by a self-adjoint operator A , corresponds to a pair of continuous functions on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of the von Neumann algebra \mathcal{R} .

In section 2, we will arrive at the definition of antonymous functions by a motivation from physics. This will give a clear physical interpretation of both antonymous and observable functions, which was lacking up to now. Antonymous and observable functions show up as the lower resp. upper integration limit in the integral defining physical expectation values. The relation between antonymous and observable functions is shown. Additionally, we give a presheaf construction leading to the definition of antonymous functions.

The properties (i)–(iii) mentioned above are demonstrated in section 3, using some results from [deG05b]. Some differences between antonymous functions and observable functions are shown.

1.1 Conventions

Throughout, \mathcal{R} denotes a unital von Neumann algebra, given as a subalgebra of the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on some suitable Hilbert space \mathcal{H} . The real linear space of self-adjoint operators in \mathcal{R} is denoted by \mathcal{R}_{sa} , $\mathcal{P}(\mathcal{R})$ is the projection lattice of \mathcal{R} . A spectral family in $\mathcal{P}(\mathcal{R})$ is a family $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ of projections in $\mathcal{P}(\mathcal{R})$ such that (i) for all $\mu < \lambda$, $E_\mu \leq E_\lambda$, (ii) $\bigwedge_{\lambda \in \mathbb{R}} E_\lambda = 0$ and (iii) $\bigvee_{\lambda \in \mathbb{R}} E_\lambda = I$. A spectral family E is bounded if there are $a, b \in \mathbb{R}$ such that $E_\lambda = 0$ for all $\lambda < a$ and $E_\lambda = I$ for all $\lambda > b$.

Let $A \in \mathcal{R}_{sa}$ be self-adjoint. The spectral theorem shows that there is a bounded spectral family $E^A = (E_\lambda^A)_{\lambda \in \mathbb{R}}$ in $\mathcal{P}(\mathcal{R})$ such that $A = \int_{-|A|}^{|A|} \lambda dE_\lambda$ (see e.g. [KadRin97]). In order to make the spectral family unique, one often additionally requires E^A to be *right-continuous*, i.e. for all $\lambda \in \mathbb{R}$,

$$\text{s-lim}_{\varepsilon \rightarrow +0} E_{\lambda+\varepsilon}^A = E_\lambda^A, \quad \bigwedge_{\mu > \lambda} E_\mu^A = E_\lambda^A.$$

We will denote right-continuous spectral families by E (or E^A) throughout. Alternatively, one can also require *left-continuity*. Let $F^A = (F_\lambda^A)_{\lambda \in \mathbb{R}}$ denote the (unique) left-continuous spectral family of A , in particular, for all $\lambda \in \mathbb{R}$,

$$\text{s-lim}_{\varepsilon \rightarrow +0} F_{\lambda-\varepsilon}^A = F_\lambda^A, \quad \bigvee_{\mu < \lambda} F_\mu^A = F_\lambda^A.$$

We will denote left-continuous spectral families by F (or F^A) throughout. Let $\mathcal{E}(\mathcal{R})$ denote the set of bounded right-continuous spectral families in $\mathcal{P}(\mathcal{R})$, and let $\mathcal{F}(\mathcal{R})$ denote the set of bounded left-continuous spectral families in $\mathcal{P}(\mathcal{R})$. We have a bijection $\varphi : \mathcal{E}(\mathcal{R}) \rightarrow \mathcal{F}(\mathcal{R})$, which is obviously defined as follows: let $E \in \mathcal{E}(\mathcal{R})$ be a right-continuous spectral family. Define a left-continuous spectral family $\varphi(E) \in \mathcal{F}(\mathcal{R})$ by

$$\forall \lambda \in \mathbb{R} : \varphi(E)_\lambda := \bigvee_{\mu < \lambda} E_\mu.$$

Conversely, let $F \in \mathcal{F}(\mathcal{R})$ be a left-continuous spectral family. Define a right-continuous spectral family $\varphi^{-1}(F) \in \mathcal{E}(\mathcal{R})$ by

$$\forall \lambda \in \mathbb{R} : \varphi^{-1}(F)_\lambda := \bigwedge_{\mu > \lambda} F_\mu.$$

Unless otherwise mentioned, spectral family will always mean left-continuous spectral family.

2 Motivation

2.1 Antonymous functions from physical expectation values

In physics, von Neumann algebras show up as the *algebras of observables* of quantum systems; the self-adjoint operators $A \in \mathcal{R}_{sa}$ are *quantum observables*.

Let $\mathcal{R} = \mathcal{L}(\mathcal{H})$. The *expectation value* of an observable $A \in \mathcal{L}(\mathcal{H})_{sa}$ in the pure state $\rho_{\mathbb{C}x} = P_{\mathbb{C}x}$ ($|x| = 1$) is given by

$$\langle Ax, x \rangle = \text{tr}(\rho_{\mathbb{C}x} A) = \int_{-|A|}^{|A|} \lambda \, d \langle E_\lambda^A x, x \rangle, \quad (1)$$

where $E^A = (E_\lambda^A)_{\lambda \in \mathbb{R}}$ is the (right-continuous) spectral family of A . (This can be found in textbooks on quantum mechanics, see e.g. [Emch84].) Let $\mathcal{Q}(\mathcal{L}(\mathcal{H}))$ be the Stone spectrum of $\mathcal{L}(\mathcal{H})$, that is, the space of maximal filter bases in the projection lattice $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. Let $\mathbb{C}x$ be the line in Hilbert space defined by x , and let $P_{\mathbb{C}x}$ be the orthogonal projection onto this line. There is a quasipoint $\mathfrak{B}_{\mathbb{C}x}$, i.e. an element of the Stone spectrum $\mathcal{Q}(\mathcal{L}(\mathcal{H}))$, given by

$$\mathfrak{B}_{\mathbb{C}x} := \{P \in \mathcal{P}(\mathcal{L}(\mathcal{H})) \mid P \geq P_{\mathbb{C}x}\}.$$

This obviously is a maximal filter base in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. Such a quasipoint $\mathfrak{B}_{\mathbb{C}x}$ is an isolated point of $\mathcal{Q}(\mathcal{L}(\mathcal{H}))$ and is called an **atomic quasipoint**. Let

$$\begin{aligned} f_A : \mathcal{Q}(\mathcal{L}(\mathcal{H})) &\longrightarrow \mathbb{R} \\ \mathfrak{B} &\longmapsto \inf\{\lambda \in \mathbb{R} \mid E_\lambda^A \in \mathfrak{B}\} \end{aligned}$$

be the observable function of A , where $E^A = (E_\lambda^A)_{\lambda \in \mathbb{R}}$ is the right-continuous spectral family of A [deG05b]. Since we have

$$\begin{aligned} f_A(\mathfrak{B}_{\mathbb{C}x}) &= \inf\{\lambda \in \mathbb{R} \mid E_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\} \\ &= \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda^A\}, \end{aligned}$$

$P_{\mathbb{C}x} \leq E_\mu^A$ holds for all $\mu > f_A(\mathfrak{B}_{\mathbb{C}x})$, so these values do not contribute to the integral and we can write

$$\langle Ax, x \rangle = \text{tr}(\rho_{\mathbb{C}x} A) = \int_{-|A|}^{f_A(\mathfrak{B}_{\mathbb{C}x})} \lambda \, d\langle E_\lambda^A x, x \rangle. \quad (2)$$

The physical meaning of the observable function f_A , evaluated at $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}(\mathcal{L}(\mathcal{H}))$, hence is the following: $f_A(\mathfrak{B}_{\mathbb{C}x})$ is the *largest possible measurement result* one can obtain for the observable A when the system is in the pure state $\rho_{\mathbb{C}x}$.

If the physical system is not in a pure state $\rho_{\mathbb{C}x}$, but in a mixed state ρ , we use the fact that such a mixed state is the convex combination of pure states, $\rho = \sum_j a_j \rho_{\mathbb{C}x_j}$. (The unit vectors x_0, x_1, \dots are chosen orthogonal.) Since the trace is linear, for all $A \in \mathcal{R}_{sa}$ we have

$$\text{tr}(\rho A) = \sum_j a_j \text{tr}(\rho_{\mathbb{C}x_j} A),$$

so the calculation of every expectation value $\text{tr}(\rho A)$ can be reduced to the simple situation (2).

This suggests to search for a similar description for the lower integration limit in (2). It can usually be taken larger than $-|A|$: we do not have to consider those values of λ for which $E_\lambda^A x = 0$ holds. Geometrically, for those λ the line $\mathbb{C}x$ is contained in the orthogonal complement of $U_{E_\lambda^A}$, the closed subspace E_λ^A projects onto. We define a function $g_A : \mathbb{P}(\mathcal{H}) \rightarrow \mathbb{R}$ on projective Hilbert space to encode this:

$$\begin{aligned} g_A(\mathbb{C}x) &:= \sup\{\lambda \in \mathbb{R} \mid \mathbb{C}x \perp U_{E_\lambda^A}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} E_\lambda^A = 0\} \\ &= \sup\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq I - E_\lambda^A\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\}. \end{aligned}$$

Let F^A denote the left-continuous spectral family of A . Since $\sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\} = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\}$, we also have

$$g_A(\mathbb{C}x) = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\}.$$

(While this is not important currently, left-continuous spectral families turn out to be more convenient in some situations.) The definition of g_A allows a natural generalization:

Definition 1 *Let \mathcal{R} be a von Neumann algebra, and let $A \in \mathcal{R}_{sa}$ be a self-adjoint operator with spectral family F^A . The **function antonymous with A** (or the **antonymous function of A**) is the function*

$$g_A : \mathcal{Q}(\mathcal{R}) \longrightarrow \mathbb{R} \\ \mathfrak{B} \longmapsto \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\}.$$

The set of antonymous functions of \mathcal{R} is denoted by $\mathcal{A}(\mathcal{R})$.

The name “antonymous function” stems from the fact that we consider the supremum of those real values λ for which $I - F_\lambda^A$ is contained in \mathfrak{B} , and $I - F_\lambda^A$ is “opposite to” or “antonymous with” F^A (and hence with A).

Just as wanted, the value $g_A(\mathfrak{B}_{\mathbb{C}x})$ obviously is the *smallest possible measurement result* one can obtain for the observable A when the system is in the pure state $\rho_{\mathbb{C}x}$ (where, as before, $|x| = 1$). We obtain for the expectation value of the observable $A \in \mathcal{R}_{sa}$ in the pure state $\rho_{\mathbb{C}x}$:

$$\langle Ax, x \rangle = \text{tr}(\rho_{\mathbb{C}x} A) = \int_{g_A(\mathfrak{B}_{\mathbb{C}x})}^{f_A(\mathfrak{B}_{\mathbb{C}x})} \lambda \, d\langle E_\lambda^A x, x \rangle. \quad (3)$$

There is a close relationship between antonymous and observable functions: If $A \in \mathcal{R}_{sa}$ is a self-adjoint operator with right-continuous spectral family E^A , then $(I - E_{(1-\lambda)-}^A)_{\lambda \in \mathbb{R}}$ is the right-continuous spectral family of the operator $I - A$. Here, $I - E_{(1-\lambda)-}^A = \bigwedge_{\mu > \lambda} (I - E_{1-\mu}^A)$. Now the observable function f_{I-A} of the operator $I - A$ is

$$\begin{aligned} f_{I-A}(\mathfrak{B}) &= \inf\{\lambda \in \mathbb{R} \mid I - E_{(1-\lambda)-}^A \in \mathfrak{B}\} \\ &= \inf\{1 - \lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}\} \\ &= 1 - \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}\} \\ &= 1 - g_A(\mathfrak{B}). \end{aligned}$$

A simple result on observable functions (lemma 2.1 in [deG05b]) shows that $1 - f_{I-A} = -f_{-A}$. Analogously, $1 - g_{I-A} = -g_{-A}$, see Lemma 11 below. It is interesting that the smallest possible measurement result $g_A(\mathfrak{B}_{\mathbb{C}x})$ and the largest possible measurement result $f_A(\mathfrak{B}_{\mathbb{C}x})$ for the observable $A \in \mathcal{R}_{sa}$ in the state $\rho_{\mathbb{C}x}$ are thus related by

$$\begin{aligned} g_A(\mathfrak{B}_{\mathbb{C}x}) &= 1 - f_{I-A}(\mathfrak{B}_{\mathbb{C}x}) = -f_{-A}(\mathfrak{B}_{\mathbb{C}x}), \\ f_A(\mathfrak{B}_{\mathbb{C}x}) &= 1 - g_{I-A}(\mathfrak{B}_{\mathbb{C}x}) = -g_{-A}(\mathfrak{B}_{\mathbb{C}x}). \end{aligned}$$

While many properties of antonymous functions can be derived from those of observable functions (and vice versa) using these relations, others are not obvious. Moreover, there is no apparent reason to consider the function $1 - f_{I-A}$, while the expression (3) for the expectation values gives a clear physical interpretation of antonymous and observable functions. The antonymous function g_A and the observable function f_A represent two different aspects of the observable A .

2.2 Antonymous functions from a presheaf construction

The definition of antonymous functions can also be motivated from a presheaf construction, as will be shown here. As before, let \mathcal{R} denote a unital von Neumann algebra, and let $\mathcal{S}(\mathcal{R})$ denote the category of subalgebras of \mathcal{R} of the form PRP , $P \in \mathcal{P}(\mathcal{R})$. A morphism $PRP \rightarrow QRQ$ between objects PRP, QRQ of $\mathcal{S}(\mathcal{R})$ exists if and only if $P \leq Q$. Then the morphism $PRP \rightarrow QRQ$ simply is the inclusion.

Definition 2 *Let \mathcal{R} be a unital von Neumann algebra. An **opposite spectral family** in $\mathcal{P}(\mathcal{R})$ is a family $G = (G_\lambda)_{\lambda \in \mathbb{R}}$ of projections in $\mathcal{P}(\mathcal{R})$, indexed by the real numbers, such that*

- (i) $\forall \lambda, \mu \in \mathbb{R}, \lambda < \mu : G_\lambda \geq G_\mu$,
- (ii) $\bigwedge_{\mu < \lambda} G_\mu = G_\lambda$,
- (iii) $\bigwedge_{\lambda \in \mathbb{R}} G_\lambda = 0, \quad \bigvee_{\lambda \in \mathbb{R}} G_\lambda = I$.

*An opposite spectral family G is called **bounded** if there exist $a, b \in \mathbb{R}$ such that*

- (iv) $\forall \lambda < a : G_\lambda = I, \quad \forall \lambda > b : G_\lambda = 0$.

Let $\mathcal{F}^o(\mathcal{R})$ denote the set of bounded opposite spectral families in $\mathcal{P}(\mathcal{R})$.

If G is a (bounded) opposite spectral family, then $I - G = (I - G_\lambda)_{\lambda \in \mathbb{R}}$ is a (bounded) spectral family. In particular, $I - G$ is left-continuous. Conversely, let $F^A \in \mathcal{F}(\mathcal{R})$ be a bounded spectral family in $\mathcal{P}(\mathcal{R})$, corresponding to a self-adjoint operator $A \in \mathcal{R}_{sa}$. Then $I - F^A = (I - F_\lambda^A)_{\lambda \in \mathbb{R}}$ is a bounded opposite spectral family. This (and the spectral theorem) shows that there are bijections

$$\mathcal{F}^o(\mathcal{R}) \simeq \mathcal{F}(\mathcal{R}) \simeq \mathcal{R}_{sa}$$

between the sets of bounded opposite spectral families in $\mathcal{P}(\mathcal{R})$, bounded spectral families in $\mathcal{P}(\mathcal{R})$ and self-adjoint operators in \mathcal{R} . Any bounded opposite spectral family $G \in \mathcal{F}^o(\mathcal{R})$ hence is of the form $G = I - F^A$ for some self-adjoint operator $A \in \mathcal{R}_{sa}$.

Let $P \leq Q$ be projections in $\mathcal{P}(\mathcal{R})$. We will now define a restriction mapping

$$\begin{aligned} \rho_P^Q : \mathcal{F}^o(QRQ) &\longrightarrow \mathcal{F}^o(PRP) \\ I - F^A &\longmapsto (I - F^A)^P \end{aligned}$$

between bounded opposite spectral families: let $I - F^A$ be a bounded opposite spectral family in $\mathcal{P}(QRQ)$ (of course, the unit I in QRQ is $I = Q$), and let

$$(I - F^A)^P = ((I - F_\lambda^A)^P)_{\lambda \in \mathbb{R}} := ((I - F_\lambda^A) \wedge P)_{\lambda \in \mathbb{R}}.$$

Lemma 3 *$(I - F^A)^P$ is a bounded opposite spectral family in $\mathcal{P}(PRP)$.*

Proof. $(I - F^A)^P$ fulfills all the defining conditions of a bounded spectral family:

- (i) for all $\lambda < \mu$, $(I - F_\lambda^A) \wedge P \geq (I - F_\mu^A) \wedge P$,
- (ii) $\bigwedge_{\mu < \lambda} ((I - F_\mu^A) \wedge P) = P \wedge \bigwedge_{\mu < \lambda} (I - F_\mu^A) = P \wedge (I - \bigvee_{\mu < \lambda} F_\mu^A) = P \wedge (I - F_\lambda^A)$,
- (iv) (implies (iii), also) $I - F^A$ is bounded, so there exist $a, b \in \mathbb{R}$ such that $I - F_\lambda^A = I$ for all $\lambda < a$ and $I - F_\lambda^A = 0$ for all $\lambda > b$. Clearly, we have $(I - F_\lambda^A) \wedge P = P$ for all $\lambda < a$ and $(I - F_\lambda^A) \wedge P = 0$ for all $\lambda > b$, so $(I - F^A)^P$ is bounded, too.

■

Corollary 4 *The sets $\mathcal{F}^o(Q\mathcal{R}Q)$, $Q \in \mathcal{P}(\mathcal{R})$ of bounded opposite spectral families, together with the restriction mappings $\rho_P^Q : \mathcal{F}^o(Q\mathcal{R}Q) \rightarrow \mathcal{F}^o(P\mathcal{R}P)$ for any $P, Q \in \mathcal{P}(\mathcal{R})$ such that $P < Q$ form a presheaf on the category $\mathcal{S}(\mathcal{R})$ of subalgebras of \mathcal{R} . It will be called the **opposite spectral presheaf** on the category $\mathcal{S}(\mathcal{R})$.*

Now let $\mathcal{R} = \mathcal{L}(\mathcal{H})$. The idea is to restrict a bounded opposite spectral family $I - F^A \subseteq \mathcal{P}(\mathcal{L}(\mathcal{H}))$ to a one-dimensional subspace: let $P = P_{\mathbb{C}x}$ be the projection onto the one-dimensional subspace $\mathbb{C}x \subset \mathcal{H}$, and let $(I - F^A) \wedge P_{\mathbb{C}x}$ be the restriction of $I - F^A$ to $P_{\mathbb{C}x}\mathcal{L}(\mathcal{H})P_{\mathbb{C}x} = \mathbb{C}P_{\mathbb{C}x}$. Then $(I - F^A) \wedge P_{\mathbb{C}x}$ is given by

$$((I - F^A) \wedge P_{\mathbb{C}x})_\lambda = (I - F_\lambda^A) \wedge P_{\mathbb{C}x} = \begin{cases} P_{\mathbb{C}x} & \text{for } \lambda \leq d(\mathbb{C}x) \\ 0 & \text{for } \lambda > d(\mathbb{C}x) \end{cases}$$

for some real parameter $d(\mathbb{C}x)$. Let $\mathfrak{B}_{\mathbb{C}x}$ be the atomic quasipoint $\mathfrak{B}_{\mathbb{C}x} = \{P \in \mathcal{P}(\mathcal{L}(\mathcal{H})) \mid P \geq P_{\mathbb{C}x}\}$. The parameter $d(\mathbb{C}x)$ is given by

$$\begin{aligned} d(\mathbb{C}x) &= \sup\{\lambda \in \mathbb{R} \mid (I - F_\lambda^A) \wedge P_{\mathbb{C}x} = P_{\mathbb{C}x}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq I - F_\lambda^A\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}_{\mathbb{C}x}\} \\ &= g_A(\mathfrak{B}_{\mathbb{C}x}), \end{aligned}$$

so the restriction of the bounded opposite spectral family $I - F^A$ to one-dimensional subspaces leads to the definition of the antonymous function g_A of $A \in \mathcal{R}_{sa}$. This is similar to the situation for observable functions, which were originally defined using the restriction of spectral families to one-dimensional subspaces [deG01]. The definition of opposite spectral families was made to capture the intuitive idea that the antonymous function g_A represents information that is contained in the “opposite” $I - F^A$ of the spectral family F^A .

3 Some properties of antonymous functions

In this section, we clarify some of the properties of antonymous functions. In particular, we will show that antonymous functions are generalized Gelfand transforms in the sense that for an abelian von Neumann algebra, the antonymous function g_A and \hat{A} , the Gelfand transform of a self-adjoint operator A , coincide.

3.1 Basic properties

In this subsection, we draw on some results found for observable functions. To make this paper reasonably self-contained, we give full proofs. Compare [deG05b], in particular, for the proofs of Props. 6, 12, Lemma 11 and Thm. 14.

We start with the important example of the antonymous function of a projection P , which turns out to be a characteristic function:

Example 5 *Let $P \in \mathcal{P}(\mathcal{R}) \setminus \{0\}$ be a non-zero projection in the von Neumann algebra \mathcal{R} . The spectral family of P is given by*

$$F_\lambda^P = \begin{cases} 0 & \text{for } \lambda \leq 0 \\ I - P & \text{for } 0 < \lambda \leq 1 \\ I & \text{for } \lambda > 1. \end{cases}$$

If $\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})$, i.e. $P \in \mathfrak{B}$, then

$$g_P(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^P \in \mathfrak{B}\} = 1,$$

while for a quasipoint $\mathfrak{B} \notin \mathcal{Q}_P(\mathcal{R})$ we have $g_P(\mathfrak{B}) = 0$, so

$$g_P = \chi_{\mathcal{Q}_P(\mathcal{R})}.$$

The antonymous function g_P of a projection P is continuous, since the sets $\mathcal{Q}_P(\mathcal{R})$ are closed-open.

(The observable function f_P of a projection P is $f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$.) The image of the antonymous function g_P is $\{0, 1\}$, and this coincides with the spectrum $\text{sp } P$ of P . (If $P = I$, then $\text{im } g_I = \text{im } \chi_{\mathcal{Q}_I(\mathcal{R})} = \text{im } \chi_{\mathcal{Q}(\mathcal{R})} = \{1\} = \text{sp } I$.) We now show that $\text{im } g_A = \text{sp } A$ holds not just for projections, but for all self-adjoint operators A :

Proposition 6 *Let \mathcal{R} be a von Neumann algebra, and let $A \in \mathcal{R}_{sa}$. Then $\text{im } g_A = \text{sp } A$.*

Proof. $\text{sp } A$ consists of those $\lambda \in \mathbb{R}$ such that F_λ^A is non-constant on every open neighbourhood of λ . Let $\lambda_0 \in \text{im } g_A$, but $\lambda_0 \notin \text{sp } A$. Then there is some $\varepsilon > 0$ such that

$$\forall \lambda \in]\lambda_0 - \varepsilon, \lambda_0 + \varepsilon[: F_\lambda^A = F_{\lambda_0}^A.$$

Let $\mathfrak{B} \in \bar{g}_A^{-1}(\lambda_0)$. From the definition of g_A , we obtain $g_A(\mathfrak{B}) \geq \lambda_0 + \varepsilon$, which is a contradiction. This implies

$$\text{im } g_A \subseteq \text{sp } A.$$

Conversely, let $\lambda_0 \in \text{sp } A$. We have to find a quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ such that $g_A(\mathfrak{B}) = \lambda_0$.

Case (i): $F_\lambda^A < F_{\lambda_0}^A$ for all $\lambda < \lambda_0$. We choose some \mathfrak{B} that contains all projections $F_{\lambda_0}^A - F_\lambda^A$ for $\lambda < \lambda_0$. Then all the projections $I - F_\lambda^A$ for $\lambda < \lambda_0$ are contained in \mathfrak{B} , since $I - F_\lambda^A \geq F_{\lambda_0}^A - F_\lambda^A$ and \mathfrak{B} is a dual ideal in $\mathcal{P}(\mathcal{R})$. Moreover, $F_{\lambda_0}^A \in \mathfrak{B}$, since

$F_{\lambda_0}^A > F_{\lambda_0}^A - F_{\lambda}^A$ for $\lambda < \lambda_0$, so $I - F_{\lambda_0}^A \notin \mathfrak{B}$ (the projections $F_{\lambda_0}^A$ and $I - F_{\lambda_0}^A$ cannot both be contained in a quasipoint \mathfrak{B} , since \mathfrak{B} is a filter base and hence does not contain the zero projection $0 = F_{\lambda_0}^A \wedge (I - F_{\lambda_0}^A)$). This implies $g_A(\mathfrak{B}) = \lambda_0$.

Case (ii) (the two cases are not mutually exclusive): there is a decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lambda_0 = \lim_{n \rightarrow \infty} \lambda_n$ and $F_{\lambda_{n+1}}^A < F_{\lambda_n}^A$ for all n . Take a quasipoint \mathfrak{B} that contains all projections $F_{\mu}^A - F_{\lambda_0}^A$ for $\mu > \lambda_0$. Then \mathfrak{B} contains $I - F_{\lambda_0}^A$ and all projections F_{μ}^A for $\mu > \lambda_0$, so $I - F_{\mu}^A \notin \mathfrak{B}$ for $\mu > \lambda_0$. ■

Let $A \in \mathcal{R}_{sa}$ be self-adjoint, f_A its observable function and g_A its antonymous function. We saw in section 2 that for the von Neumann algebra $\mathcal{R} = \mathcal{L}(\mathcal{H})$, $g_A(\mathfrak{B}_{\mathbb{C}x}) \leq f_A(\mathfrak{B}_{\mathbb{C}x})$ holds for all atomic quasipoints $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}(\mathcal{L}(\mathcal{H}))$. Not surprisingly, this also holds more generally:

Proposition 7 *Let \mathcal{R} be a von Neumann algebra, and let $A \in \mathcal{R}_{sa}$ with antonymous function g_A and observable function f_A . For all quasipoints $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, we have $g_A(\mathfrak{B}) \leq f_A(\mathfrak{B})$.*

Proof. Let E^A be the right-continuous spectral family of A , and let F^A be the left-continuous one. Then we have

$$\begin{aligned} g_A(\mathfrak{B}) &= \sup\{\lambda \in \mathbb{R} \mid I - F_{\lambda}^A \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - E_{\lambda}^A \in \mathfrak{B}\}. \end{aligned}$$

Assume that there is some quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ such that $g_A(\mathfrak{B}) > f_A(\mathfrak{B})$. Choose some $\lambda_0 \in]f_A(\mathfrak{B}), g_A(\mathfrak{B})[$. Then we have $g_A(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - E_{\lambda}^A \in \mathfrak{B}\} > \lambda_0$, so the quasipoint \mathfrak{B} must contain a projection $I - E_{\lambda_1}^A$ for $\lambda_1 > \lambda_0$. E^A is a spectral family, so $I - E_{\lambda_1}^A \leq I - E_{\lambda_0}^A$. Since \mathfrak{B} is a dual ideal in $\mathcal{P}(\mathcal{R})$, we have $I - E_{\lambda_0}^A \in \mathfrak{B}$.

On the other hand, $f_A(\mathfrak{B}) = \inf\{\lambda \in \mathbb{R} \mid E_{\lambda}^A \in \mathfrak{B}\} < \lambda_0$, so \mathfrak{B} contains a projection $E_{\lambda_2}^A$ for $\lambda_2 < \lambda_0$. $E_{\lambda_2}^A \leq E_{\lambda_0}^A$ holds, so $E_{\lambda_0}^A \in \mathfrak{B}$. But $I - E_{\lambda_0}^A, E_{\lambda_0}^A$ cannot both be contained in a quasipoint \mathfrak{B} , since \mathfrak{B} is a filter base. ■

Example 8 *Let \mathcal{R} be a non-abelian von Neumann algebra. If $P \in \mathcal{P}(\mathcal{R})$ is a projection, then the strict inequality $g_P(\mathfrak{B}) < f_P(\mathfrak{B})$ holds for all quasipoints \mathfrak{B} neither contained in $\mathcal{Q}_P(\mathcal{R})$ nor in $\mathcal{Q}_{I-P}(\mathcal{R})$. Namely, $g_P(\mathfrak{B}) = 0 < 1 = f_P(\mathfrak{B})$ for such a quasipoint.*

While for non-abelian algebras \mathcal{R} , there are quasipoints neither containing a given projection P nor $I - P$, the next lemma, which is a variant of a lemma from [deG01], shows that this cannot happen for abelian von Neumann algebras:

Lemma 9 *Let \mathcal{R} be an abelian von Neumann algebra, and let $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ be a quasipoint of \mathcal{R} . For all projections $P \in \mathcal{P}(\mathcal{R})$, either $P \in \mathfrak{B}$ or $I - P \in \mathfrak{B}$.*

Proof. Clearly, at most one of the projections $P, I - P$ can be contained in a quasipoint \mathfrak{B} , since \mathfrak{B} is a filter base. In order to show that at least one of the projections $P, I - P$

is contained in \mathfrak{B} , we use the fact that the projection lattice $\mathcal{P}(\mathcal{R})$ of \mathcal{R} is distributive, since \mathcal{R} is abelian. $P, (I - P) \notin \mathfrak{B}$ would imply

$$\exists Q, R \in \mathfrak{B} : P \wedge Q = (I - P) \wedge R = 0,$$

so

$$\begin{aligned} Q \wedge R &= (P \vee (I - P)) \wedge Q \wedge R \\ &= (P \wedge Q \wedge R) \vee ((I - P) \wedge Q \wedge R) = 0, \end{aligned}$$

contradicting $Q \wedge R \in \mathfrak{B}$. ■

In other words, for an abelian von Neumann algebra \mathcal{R} , a quasipoint \mathfrak{B} is an *ultrafilter* in the projection lattice $\mathcal{P}(\mathcal{R})$.

The next proposition shows that for an abelian algebra \mathcal{R} , the antonymous function g_A and the observable function f_A of a self-adjoint operator $A \in \mathcal{R}$ coincide:

Proposition 10 *Let \mathcal{R} be an abelian von Neumann algebra, and let $A \in \mathcal{R}_{sa}$ with antonymous function g_A and observable function f_A . Then $g_A = f_A$.*

Proof. Let F^A be the left-continuous spectral family of A and E^A the right-continuous one. Let $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ be a quasipoint. We have

$$\begin{aligned} g_A(\mathfrak{B}) &= \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}\}. \end{aligned}$$

According to Lemma 9, for all $P \in \mathcal{P}(\mathcal{R})$, either $P \in \mathfrak{B}$ or $I - P \in \mathfrak{B}$. Since $I - E_\lambda^A \notin \mathfrak{B}$ for all $\lambda > g_A(\mathfrak{B})$, we have $E_\lambda^A \in \mathfrak{B}$ for all $\lambda > g_A(\mathfrak{B})$ (and $E_\mu^A \notin \mathfrak{B}$ for all $\mu < g_A(\mathfrak{B})$) and thus

$$f_A(\mathfrak{B}) = \inf\{\lambda \in \mathbb{R} \mid E_\lambda^A \in \mathfrak{B}\} = g_A(\mathfrak{B}).$$

■

Lemma 11 *Let $A \in \mathcal{R}_{sa}$, and let $t \in \mathbb{R}$. Then $g_{A+tI} = t + g_A$.*

Proof. For all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, we have

$$\begin{aligned} g_{A+tI}(\mathfrak{B}) &= \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^{A+tI} \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - F_{\lambda-t}^A \in \mathfrak{B}\} \\ &= \sup\{t + \lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\} \\ &= t + \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\} \\ &= t + g_A(\mathfrak{B}). \end{aligned}$$

■

We will now show that the antonymous function g_A of a finite real linear combination $A = \sum_{j=1}^n a_j P_j$ of pairwise orthogonal non-zero projections $P_1, \dots, P_n \in \mathcal{P}(\mathcal{R})$ with non-zero real coefficients $a_1 \leq \dots \leq a_n$ is a step function and hence continuous. Let $P := \sum_{j=1}^n P_j$. 0 is contained in the spectrum of the operator A if and only if $P \neq I$. The following discussion is only needed in the case $P \neq I$, but things also work trivially if $P = I$.

Let k_0 be the greatest index such that $a_{k_0} < 0$. If there is no $a_k < 0$, then $k_0 := 0$. For $k = 1, \dots, n+1$, define

$$b_k := \begin{cases} a_k & \text{for } k \leq k_0 \\ 0 & \text{for } k_0 + 1 \\ a_{k-1} & \text{for } k > k_0 + 1, \end{cases}$$

$$Q_k := \begin{cases} P_k & \text{for } k \leq k_0 \\ I - P & \text{for } k = k_0 + 1 \\ P_{k-1} & \text{for } k > k_0 + 1. \end{cases}$$

Then $\sum_{k=1}^{n+1} Q_k = I$ and $A = \sum_{k=1}^{n+1} b_k Q_k$, where $b_1 \leq \dots \leq b_{n+1}$ and $b_{k_0+1} = 0$, but $Q_{k_0+1} \neq 0$ unless $P = I$. Thus the spectral family F^A of A is:

$$F_\lambda^A = \begin{cases} 0 & \text{for } \lambda \leq b_1 \\ Q_1 + \dots + Q_k & \text{for } b_k < \lambda \leq b_{k+1} \text{ } (k = 1, \dots, n) \\ I & \text{for } \lambda > b_{n+1}. \end{cases}$$

Let $Q_0 := 0$. Each quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is contained in exactly one of the pairwise disjoint sets

$$\begin{aligned} M_1 &:= \mathcal{Q}_{I-Q_0}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+Q_1)}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+Q_1)}(\mathcal{R}), \\ M_2 &:= \mathcal{Q}_{I-(Q_0+Q_1)}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+Q_1+Q_2)}(\mathcal{R}), \\ M_3 &:= \mathcal{Q}_{I-(Q_0+Q_1+Q_2)}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+\dots+Q_3)}(\mathcal{R}), \\ &\vdots \\ &\vdots \\ &\vdots \\ M_{n+1} &:= \mathcal{Q}_{I-(Q_0+\dots+Q_n)}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+\dots+Q_{n+1})}(\mathcal{R}) \\ &= \mathcal{Q}_{I-(Q_0+Q_1+\dots+Q_n)}(\mathcal{R}). \end{aligned}$$

The last equality holds since $Q_0 + Q_1 + \dots + Q_{n+1} = I$, so $\mathcal{Q}_{I-(Q_0+Q_1+\dots+Q_{n+1})}(\mathcal{R}) = \mathcal{Q}_0(\mathcal{R}) = \emptyset$. Now, for $k = 1, \dots, n+1$, we have

$$\begin{aligned} \mathfrak{B} \in M_k &\implies g_A(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A = I - (Q_0 + Q_1 + \dots + Q_{k-1})\} \\ &= \sup\{\lambda \in \mathbb{R} \mid F_\lambda^A = Q_0 + Q_1 + \dots + Q_{k-1}\} \\ &= b_k. \end{aligned}$$

Thus we obtain

Proposition 12 *Let $P_1, P_2, \dots, P_n \in \mathcal{P}(\mathcal{R})$ be pairwise orthogonal non-zero projections and $A := \sum_{j=1}^n a_j P_j$ with real non-zero coefficients $a_1 \leq \dots \leq a_n$. We write $A = \sum_{k=1}^{n+1} b_k Q_k$ with the b_k and Q_k defined as above. Let $Q_0 := 0$. Then the antonymous function of A is given by*

$$g_A = \sum_{k=1}^{n+1} b_k \chi_{M_k} = \sum_{k=1}^{n+1} b_k \chi_{\mathcal{Q}_{I-(Q_0+\dots+Q_{k-1})}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+\dots+Q_k)}(\mathcal{R})}. \quad (*)$$

In particular, the antonymous function g_A is continuous.

The antonymous function g_A of a finite real linear combination $A := \sum_{j=1}^n a_j P_j$ of pairwise orthogonal projections hence is a step function. Of course, the coefficient b_{k_0+1} is 0, so the summand for $k_0 + 1$ can be left out from the sum (*). However, we must use the projections Q_k and not simply the P_j in (*), since they show up in the spectral family F^A . (This is not necessary if $P = \sum_{j=1}^n P_j = I$).

If we take $A = P \in \mathcal{P}(\mathcal{R})$, we get back the simple characteristic function of example 5: we have $A = 0 \cdot (I - P) + 1 \cdot P = \sum_{k=1}^2 b_k Q_k$, where $b_1 = 0$, $b_2 = 1$, $Q_1 = I - P$, $Q_2 = P$ and $Q_0 := 0$. Inserting into (*) gives

$$\begin{aligned} g_P &= b_1 \chi_{\mathcal{Q}_{I-Q_0}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+Q_1)}(\mathcal{R})} + b_2 \chi_{\mathcal{Q}_{I-(Q_0+Q_1)}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+Q_1+Q_2)}(\mathcal{R})} \\ &= b_2 \chi_{\mathcal{Q}_{I-Q_1}(\mathcal{R}) \setminus \mathcal{Q}_0(\mathcal{R})} \\ &= \chi_{\mathcal{Q}_P(\mathcal{R})}. \end{aligned}$$

Theorem 13 *Let \mathcal{R} be an abelian von Neumann algebra, and let $A = \sum_{j=1}^n a_j P_j$ for pairwise orthogonal non-zero projections P_1, \dots, P_n and non-zero real coefficients a_1, \dots, a_n . Let b_k, Q_k ($k = 1, \dots, n+1$) be defined as in Prop. 12, so $A = \sum_{k=1}^{n+1} b_k Q_k$. Then*

$$g_A = \sum_{k=1}^{n+1} b_k \chi_{\mathcal{Q}_{Q_k}(\mathcal{R})} = f_A,$$

where g_A is the antonymous function and f_A is the observable function of A .

Proof. The second equality is shown in [deG05b] (In fact, there is a minor mistake in [deG05b]: in prop. 2.6 and cor. 2.1, the projections P_j and not the Q_k are used in the expression for f_A , which is wrong unless $P = \sum_{k=1}^n P_k = I$.) The equality of g_A and f_A is shown in Prop. 10.

It might also be instructive to prove the first equality: if \mathcal{R} is abelian, then $\mathcal{P}(\mathcal{R})$ is distributive. Let $A = \sum_{k=1}^{n+1} b_k Q_k$ with b_k, Q_k as in Prop. 12. Let $Q_0 := 0$. According to Lemma 9, a quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ that does not contain $I - (Q_0 + \dots + Q_k)$ contains $Q_0 + \dots + Q_k$. Thus a quasipoint $\mathfrak{B} \in \mathcal{Q}_{I-(Q_0+\dots+Q_{k-1})}(\mathcal{R}) \setminus \mathcal{Q}_{I-(Q_0+\dots+Q_k)}(\mathcal{R})$ contains $I - (Q_0 + \dots + Q_{k-1})$ and $Q_0 + \dots + Q_k$ and hence

$$I - (Q_0 + \dots + Q_{k-1}) \wedge (Q_0 + \dots + Q_k) = Q_k \in \mathfrak{B}.$$

Conversely, even in the non-distributive case each $\mathfrak{B} \in \mathcal{Q}_{Q_k}(\mathcal{R})$ contains the projection $I - (Q_0 + \dots + Q_{k-1})$, but not the projection $I - (Q_0 + \dots + Q_k)$. So for abelian \mathcal{R} , we have

$$\chi_{\mathcal{Q}_{I-(Q_0+\dots+Q_{k-1})}(\mathcal{R})} \setminus \mathcal{Q}_{I-(Q_0+\dots+Q_k)}(\mathcal{R}) = \chi_{\mathcal{Q}_{Q_k}(\mathcal{R})}.$$

■

In order to show that the antonymous function g_A of an arbitrary self-adjoint operator $A \in \mathcal{R}_{sa}$ is continuous, we approximate g_A uniformly by the step functions from Prop. 12.

Theorem 14 *Let \mathcal{R} be an arbitrary unital von Neumann algebra, and let $A \in \mathcal{R}_{sa}$ be self-adjoint. Then the antonymous function $g_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ of A is continuous.*

Proof. Let $m := \min \operatorname{sp} A$, $M := \max \operatorname{sp} A$ and $\varepsilon > 0$. Choose $\lambda_0 \in]m - \varepsilon, m[$, $\lambda_n \in]M, M + \varepsilon[$, and $\lambda_1, \dots, \lambda_{n-1} \in]a, b[$ such that $\lambda_{k-1} < \lambda_k$ and $\lambda_k - \lambda_{k-1} < \varepsilon$ for all $k = 1, \dots, n$. Moreover, choose $a_k \in]\lambda_{k-1}, \lambda_k[$ for $k = 1, \dots, n$ and define

$$A_\varepsilon := \sum_{k=1}^n a_k (F_{\lambda_k}^A - F_{\lambda_{k-1}}^A) =: \sum_{k=1}^n a_k P_k,$$

where $P_k := F_{\lambda_k}^A - F_{\lambda_{k-1}}^A$. Let $P_0 := 0$. The spectral family of A_ε is given by

$$F_\lambda^{A_\varepsilon} = \begin{cases} 0 = F_{\lambda_0}^A & \text{for } \lambda \leq a_1 = \min \operatorname{sp} A_\varepsilon \\ F_{\lambda_k}^A & \text{for } a_k < \lambda \leq a_{k+1} \ (k = 1, \dots, n-1) \\ I = F_{\lambda_n}^A & \text{for } \lambda > a_n = \max \operatorname{sp} A_\varepsilon, \end{cases}$$

and from Prop. 12, the antonymous function of g_{A_ε} is³

$$\begin{aligned} g_{A_\varepsilon} &= \sum_{k=1}^n a_k \chi_{\mathcal{Q}_{I-(P_0+\dots+P_{k-1})}(\mathcal{R})} \setminus \mathcal{Q}_{I-(P_0+\dots+P_k)}(\mathcal{R}) \\ &= \sum_{k=1}^n a_k \chi_{\mathcal{Q}_{I-F_{\lambda_{k-1}}^A}(\mathcal{R})} \setminus \mathcal{Q}_{I-F_{\lambda_k}^A}(\mathcal{R}). \end{aligned}$$

Each quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is contained in exactly one of the pairwise disjoint sets

$$\begin{aligned} N_1 &:= \mathcal{Q}_{I-F_{\lambda_0}^A}(\mathcal{R}) \setminus \mathcal{Q}_{I-F_{\lambda_1}^A}(\mathcal{R}) = \mathcal{Q}(\mathcal{R}) \setminus \mathcal{Q}_{I-F_{\lambda_1}^A}(\mathcal{R}), \\ N_2 &:= \mathcal{Q}_{I-F_{\lambda_1}^A}(\mathcal{R}) \setminus \mathcal{Q}_{I-F_{\lambda_2}^A}(\mathcal{R}), \\ &\vdots \\ N_n &:= \mathcal{Q}_{I-F_{\lambda_{n-1}}^A}(\mathcal{R}) \setminus \mathcal{Q}_{I-F_{\lambda_n}^A}(\mathcal{R}) = \mathcal{Q}_{I-F_{\lambda_{n-1}}^A}(\mathcal{R}) \setminus \mathcal{Q}_0(\mathcal{R}) \\ &= \mathcal{Q}_{I-F_{\lambda_{n-1}}^A}(\mathcal{R}), \end{aligned}$$

³Since $F_{\lambda_0}^A = 0$ and $F_{\lambda_n}^A = I$, we have $\sum_{k=1}^n P_k = I$ by construction. Hence we do not have to mind if 0 is a spectral value of A or not and can skip the definition of the b_k and Q_k ($k = 1, \dots, n+1$) normally used in Prop. 12.

so, for $k = 1, \dots, n$, we obtain

$$\begin{aligned}\mathfrak{B} \in N_k &\implies g_{A_\varepsilon}(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^{A_\varepsilon} \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^{A_\varepsilon} = I - F_{\lambda_{k-1}}^A\} \\ &= a_k.\end{aligned}$$

Moreover,

$$\mathfrak{B} \in N_k \implies g_A(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - F_\lambda^A \in \mathfrak{B}\} \in [\lambda_{k-1}, \lambda_k].$$

For all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, we thus have $|(g_{A_\varepsilon} - g_A)(\mathfrak{B})| < \varepsilon$, i.e.

$$|g_{A_\varepsilon} - g_A|_\infty \leq \varepsilon,$$

so the antonymous function g_A of $A \in \mathcal{R}_{sa}$ is continuous. ■

Let $\mathcal{A}(\mathcal{R}) := \{g_A \mid A \in \mathcal{R}_{sa}\}$ denote the set of antonymous functions of \mathcal{R} , and let $\mathcal{O}(\mathcal{R}) := \{f_A \mid A \in \mathcal{R}_{sa}\}$ denote the set of observable functions of \mathcal{R} . We have shown that the set $\mathcal{A}(\mathcal{R})$ of antonymous functions is a subset of $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, the bounded continuous real-valued functions on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of \mathcal{R} . $\mathcal{A}(\mathcal{R})$ separates the points of $\mathcal{Q}(\mathcal{R})$, since $g_P = \chi_{\mathcal{Q}_P(\mathcal{R})}$ for a projection $P \in \mathcal{P}(\mathcal{R})$. $\mathcal{A}(\mathcal{R})$ contains the constant functions (choose $A = 0$ in Lemma 11). Since $g_A = 1 - f_{I-A} = -f_{-A}$ (see section 2), we have $\mathcal{A}(\mathcal{R}) = -\mathcal{O}(\mathcal{R})$.

3.2 Antonymous functions as generalized Gelfand transforms

Let \mathcal{R} be an abelian von Neumann algebra. Thm. 2.9 in [deG05b] shows that the mapping

$$\begin{aligned}\omega : \mathcal{R}_{sa} &\longrightarrow \mathcal{O}(\mathcal{R}) \\ A &\longmapsto f_A,\end{aligned}$$

sending a self-adjoint operator A to its observable function f_A , is the restriction of the Gelfand transformation to the self-adjoint operators in \mathcal{R} . Here, the Gelfand spectrum $\Omega(\mathcal{R})$ and the Stone spectrum $\mathcal{Q}(\mathcal{R})$ of \mathcal{R} are identified using thm. 3.2 of [deG05]. Let

$$\theta : \mathcal{Q}(\mathcal{R}) \longrightarrow \Omega(\mathcal{R})$$

denote the homeomorphism between the Stone spectrum and the Gelfand spectrum of \mathcal{R} . Since the Gelfand spectrum $\Omega(\mathcal{R})$ is compact, so is the Stone spectrum $\mathcal{Q}(\mathcal{R})$, and all continuous functions on $\Omega(\mathcal{R})$ (resp. $\mathcal{Q}(\mathcal{R})$) are bounded.

Let $\Gamma : \mathcal{R} \rightarrow C(\Omega(\mathcal{R}))$ denote the Gelfand transformation. This is an isometric $*$ -isomorphism, so in particular, $\Gamma(\mathcal{R}_{sa}) = C(\Omega(\mathcal{R}), \mathbb{R})$, i.e. the self-adjoint operators in \mathcal{R}

are mapped bijectively to the real-valued continuous functions on the Gelfand spectrum $\Omega(\mathcal{R})$. The homeomorphism θ from $\Omega(\mathcal{R})$ onto $\mathcal{Q}(\mathcal{R})$ induces a $*$ -isomorphism

$$\begin{aligned}\theta^* : C(\Omega(\mathcal{R})) &\longrightarrow C(\mathcal{Q}(\mathcal{R})) \\ h &\longmapsto h \circ \theta.\end{aligned}$$

Prop. 10 shows that for all $A \in \mathcal{R}_{sa}$, the antonymous function g_A coincides with the observable function f_A if \mathcal{R} is abelian. Hence we get:

Theorem 15 *Let \mathcal{R} be an abelian von Neumann algebra. Then the mapping*

$$\begin{aligned}\alpha : \mathcal{R}_{sa} &\longrightarrow \mathcal{A}(\mathcal{R}) \subseteq C(\mathcal{Q}(\mathcal{R}), \mathbb{R}), \\ A &\longmapsto g_A,\end{aligned}$$

from \mathcal{R}_{sa} to $C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, sending a self-adjoint operator A to its antonymous function g_A , is the restriction of the Gelfand transformation Γ to \mathcal{R}_{sa} . Here, the $$ -isomorphism θ^* is used to identify the homeomorphic spaces $\Gamma(\mathcal{R}_{sa}) = C(\Omega(\mathcal{R}), \mathbb{R})$ and $C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$.*

Corollary 16 *Let \mathcal{R} be an abelian von Neumann algebra. Then the mapping $\alpha : \mathcal{R}_{sa} \rightarrow C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, $A \mapsto g_A$, is surjective, i.e. every real-valued continuous function on the Stone spectrum $\mathcal{Q}(\mathcal{R})$ is an antonymous function.*

Proof. Since $\theta^* : C(\Omega(\mathcal{R})) \rightarrow C(\mathcal{Q}(\mathcal{R}))$ is a $*$ -isomorphism, we have $C(\mathcal{Q}(\mathcal{R}), \mathbb{R}) = \theta^*(C(\Omega(\mathcal{R}), \mathbb{R})) = \theta^*(\Gamma(\mathcal{R}_{sa}))$, i.e. every function $\varphi \in C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$ is of the form $\theta^*(\hat{A})$, where $\hat{A} = \Gamma(A)$ is the Gelfand transform of some self-adjoint operator $A \in \mathcal{R}_{sa}$. ■

Remark 17 *Of course, the mappings ω and α can also be considered for arbitrary, non-abelian von Neumann algebras. Then these mappings are two different non-commutative generalizations of the Gelfand transformation (for self-adjoint elements).*

Let \mathcal{R} be an arbitrary von Neumann algebra. Since every operator $B \in \mathcal{R}$ has a unique decomposition

$$B = A_1 + iA_2,$$

where A_1, A_2 are self-adjoint operators in \mathcal{R} , the mapping

$$\begin{aligned}\alpha : \mathcal{R}_{sa} &\longrightarrow C(\mathcal{Q}(\mathcal{R}), \mathbb{R}) \\ A &\longmapsto g_A\end{aligned}$$

can be extended canonically to a mapping

$$\begin{aligned}\alpha' : \mathcal{R} &\longrightarrow C(\mathcal{Q}(\mathcal{R})) \\ B &\longmapsto g_{A_1} + ig_{A_2}.\end{aligned}$$

If \mathcal{R} is abelian, then this mapping is the Gelfand transformation of \mathcal{R} (using the identification of the Stone spectrum and the Gelfand spectrum).

Addition and multiplication of antonymous functions can be defined pointwise, and thus $\mathcal{A}(\mathcal{R})$ becomes a \mathbb{R} -linear space. But of course, for a non-abelian von Neumann algebra \mathcal{R} , the mapping $\alpha : \mathcal{R}_{sa} \longrightarrow \mathcal{A}(\mathcal{R}) \subseteq C(\mathcal{Q}(\mathcal{R}), \mathbb{R})$ does not respect the linear structure of \mathcal{R}_{sa} . This mapping is \mathbb{R} -homogeneous, but we have $g_{A_1+A_2} \neq g_{A_1} + g_{A_2}$ in general ($A_1, A_2 \in \mathcal{R}_{sa}$). Consequently, the mapping $\alpha' : \mathcal{R} \longrightarrow C(\mathcal{Q}(\mathcal{R}))$, $\alpha(B) = g_{A_1} + ig_{A_2}$ is *not* an algebra homomorphism unless \mathcal{R} is abelian.

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